A Tutorial Solution of

Electromagnetic Fields and waves

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Vector Analysis

Tutorial Sheet No:1

1) Given the two vectors, $r_A = -\mathbf{a_x}-3\mathbf{a_y}-4\mathbf{a_z}$ and $r_B = 2\mathbf{a_x}+2\mathbf{a_y}+2\mathbf{a_z}$ and point C(1,3,4). Find:

   a) $\mathbf{R_{AB}}$
   b) $|\mathbf{r_A}|$
   c) $\mathbf{a_A}$
   d) $\mathbf{a_{AB}}$
   e) a unit vector directed from C towards A

Solution:
We know that,
$\mathbf{R_{AB}} = \mathbf{r_B} - \mathbf{r_A} = (2\mathbf{a_x}+2a_y+2\mathbf{a_z}) - (-\mathbf{a_x} -3\mathbf{a_y} -4\mathbf{a_z})$
Therefore, $\mathbf{R_{AB}} = 3\mathbf{a_x}+5\mathbf{a_y}+6\mathbf{a_z}$

$|\mathbf{r_A}| = \sqrt{((-1)^2 + (-3)^2 + (-4)^2)} = \sqrt{26}$ units.

$\mathbf{a_A} = \frac{r_A}{|r_A|} = -0.1961\ \mathbf{a_x}-0.588\ \mathbf{a_y}-0.784\ \mathbf{a_z}$

$\mathbf{a_{AB}} = ?$

$|\mathbf{R_{AB}}| = \sqrt{(3^2 + 5^2 + 6^2)} = 8.366$

Therefore,
$\mathbf{a_{AB}} = \frac{R_{AB}}{|R_{AB}|} = \frac{3\,a_x + 5\,a_y + 6\,a_z}{8.366}$

Therefore,
$\mathbf{a_{AB}} = 0.359\ \mathbf{a_x} + 0.598\ \mathbf{a_y} + 0.717\ \mathbf{a_z}$

$\mathbf{a_{CA}} = ?$

position vector of C is
$\mathbf{r_C} = \mathbf{a_x} + 3\ \mathbf{a_y} + 4\ \mathbf{a_z}$
Therefore, $\mathbf{R_{CA}} = \mathbf{r_A} - \mathbf{r_C} = (-\ \mathbf{a_x} -3\mathbf{a_y} - 4\mathbf{a_z}) - (\mathbf{a_x} +3\mathbf{a_y} +4\mathbf{a_z})$
$\mathbf{R_{CA}} = -2\mathbf{a_x} -6\mathbf{a_y} -8\mathbf{a_z}$
Now,
$|\mathbf{R_{CA}}| = \sqrt{((-2)^2 + (-6)^2 + (-8)^2)} = 10.198$

Therefore,
$\mathbf{a_{CA}} = \frac{R_{CA}}{|R_{CA}|} = \frac{-2\,a_x - 6\,a_y - 8\,a_z}{10.198}$

Therefore, $\mathbf{a}_{CA} = -0.196\ \mathbf{a}_x - 0.588\ \mathbf{a}_y - 0.784\ \mathbf{a}_z$

2) Given the vector field, $\mathbf{F}=0.4(y-2x)\mathbf{a}_x - [200/(x^2+y^2+z^2)]\mathbf{a}_z$.
   a. evaluate $|\mathbf{F}|$ at P(-4,3,5)
   b. find a unit vector specifying the direction of **F** at P. Describe the locus of all points for which :
   c. $|F_x| = 1$
   d. $|F_z| = 2$.

Solution:
Here, **F** at P(-4,3,5) is
$\mathbf{F} = 0.4[3-2(-4)]\mathbf{a}_x - [200/(x^2+y^2+z^2)]\mathbf{a}_z$
Therefore,
$\mathbf{F} = 4.4\mathbf{a}_x - 4\mathbf{a}_z$
So, $|\mathbf{F}| = \sqrt{(4.4)^2 + (-4)^2}$
Or, $|\mathbf{F}| = 5.95$

$\mathbf{a}_F$ at P

We have
$\mathbf{A}_F = \dfrac{F}{|F|}$
Therefore,
$\mathbf{A}_F = 0.74\mathbf{a}_x - 0.672\mathbf{a}_z$

$F_x = 1$, locus of points =?

Here,
$F_x = 0.4(y-2x)\mathbf{a}_x$
So, $|\mathbf{F}_x| = \sqrt{0.4(y-2x)}$
Or, $1 = \sqrt{0.4(y-2x)}$
Squaring both sides,
Or, $0.4(y-2x) = 1$
Or, $y - 2x = 2.5$
Therefore, $y=2x+2.5$ is the locus of points.

$|F_Z| = 2$, locus of points = ?

Here, $\mathbf{F}_Z = -[200/(x^2+y^2+z^2)]\mathbf{a}_z$
Therefore, $|\mathbf{F}_Z| = \sqrt{[200/(x^2+y^2+z^2)]^2}$
squaring both sides,
$$\frac{200}{x^2+y^2+z^2} = 2$$
$(x^2+y^2+z^2) = 100$
This is the required locus of the points.

3) Given points A(2,5,-1), B(3,-2,4) and C(-2,3,1). Find :

   a. $\mathbf{R}_{AB} . \mathbf{R}_{AC}$
   b. The angle between $\mathbf{R}_{AB}$ and $\mathbf{R}_{AC}$
   c. The length of the projection of $\mathbf{R}_{AB}$ and R_{AC}
   d. The vector projection of $\mathbf{R}_{AB}$ and $\mathbf{R}_{AC}$

Solution:

$\mathbf{R}_{AB}$=(3-2)$\mathbf{a}_x$ +(-2-5)$\mathbf{a}_y$ +(4-(-1))$\mathbf{a}_z$

$\mathbf{R}_{AB}$=$\mathbf{a}_x$ -7$\mathbf{a}_y$ +5$\mathbf{a}_z$

$\mathbf{R}_{AC}$=-4$\mathbf{a}_x$ -2$\mathbf{a}_y$ +2$\mathbf{a}_z$

Angle between $\mathbf{R}_{AB}$ and $\mathbf{R}_{AC}$

We have $\mathbf{R}_{AB} . \mathbf{R}_{AC}$ = -4+14+10 = 20

Also, $\mathbf{R}_{AB} . \mathbf{R}_{AC} = |\mathbf{R}_{AB}|\,|\mathbf{R}_{AC}| \cos\theta$ where θ is angle between R_{AB} and R_{AC}

Or, $\cos\theta = \dfrac{20}{8.66 * 4.9} = 0.471$

Therefore, $\theta = 61.87^0$

The length of the projection of $\mathbf{R}_{AB}$ on $\mathbf{R}_{AC}$

Here, $|\mathbf{R}_{AC}| = \sqrt{(-4)^2 + (-2)^2 + 2^2}$ = 4.9 units.

So, $\mathbf{a}_{AC} = \dfrac{R_{AC}}{|R_{AC}|}$ = -0.816$\mathbf{a}_x$ -0.408$\mathbf{a}_y$ +0.408$\mathbf{a}_z$

Now, projection of $\mathbf{R}_{AB}$ on $\mathbf{R}_{AC}$ is

$\mathbf{R}_P = |\mathbf{R}_{AB}| \cos\theta . 1 = R_{AB} . a_{AC}$

Or, $\mathbf{R}_P$ = (-1)*0.816 + 7*0.408 +5*0.408

Or, $\mathbf{R}_P$ = 4.08

Vector projection of $\mathbf{R}_{AB}$ on $\mathbf{R}_{AC}$ is

$\mathbf{R}_P = R_P . \mathbf{a}_{AC}$

Or, $\mathbf{R}_P$ = -3.33$\mathbf{a}_x$ -1.664$\mathbf{a}_y$ +1.664$\mathbf{a}_z$

4) A triangle is defined by the three points, A(2,-5,1) , B(-3,2,4) and C(0,3,1) . Find :
   a. $\mathbf{R_{BC}} * \mathbf{R_{BA}}$
   b. The area of the triangle
   c. A unit vector perpendicular to the plane in which the triangle is located.

Solution:
$\mathbf{R_{BC}} = 3\mathbf{a_x} + \mathbf{a_y} - 3\mathbf{a_z}$
$\mathbf{R_{BA}} = 5\mathbf{a_x} - 7\mathbf{a_y} - 3\mathbf{a_z}$

$\mathbf{R_{BC}} * \mathbf{R_{BA}} = \mathbf{a_x}\{1(-3) - (-7)(-3)\} - \mathbf{a_y}\{3(-3) - (5)(-3)\} + \mathbf{a_z}\{3(-7) - 5*1\}$
Therefore, $\mathbf{R_{BC}} * \mathbf{R_{BA}} = -24\mathbf{a_x} - 6\mathbf{a_y} + 26\mathbf{a_z}$

We know that $|\mathbf{R_{BC}} * \mathbf{R_{BA}}|$ gives the area of the parallelogram having two sides R_{BC} and R_{BA}. So ,the area of the triangle having two same sides is given by –
Area = $\frac{1}{2}(|\mathbf{R_{BC}} * \mathbf{R_{BA}}|) = \frac{1}{2}(\sqrt{24^2 + 6^2 + 26^2})$
Hence, Area of triangle = 17.944 unit^2

Here, let $\mathbf{a_R}$ is the unit vector perpendicular to the plane in which triangle lies.

Also, the vector $\mathbf{R_{BC}} * \mathbf{R_{BA}}$ is the perpendicular vector to the plane containing triangle.
Therefore, $\mathbf{a_R} = \dfrac{R_{BC} * R_{BA}}{|R_{BC} * R_{BA}|} = \dfrac{-24a_x - 6a_y + 26a_z}{35.889}$
Hence, $\mathbf{a_R} = -0.6687\mathbf{a_x} - 0.167\mathbf{a_y} + 0.724\mathbf{a_z}$
This is the required unit vector perpendicular to the plane containing the given triangle.

5) Given P($\rho = 6$, $\Phi = 125^0$, $z = -3$) and Q(x=3, y=-1, z=4) . Find the distance from :

a. P to the origin
b. Q perpendicularly to the Z-axis
c. P to Q

Solution:

Here, P($\rho = 6$, $\Phi = 125^0$, $z = -3$)

We have,

$x = \rho \cos\Phi = 6 \cos 125 = -3.441$

$y = \rho \sin\Phi = 6 \sin 125 = 4.915$

Therefore, point P can be rewritten as P(-3.441, 4.915, -3)

So, distance OP = $\sqrt{3.441^2 + 4.915^2 + 3^2}$

Or, OP = 6.708 units.

QR = ?

We know the point Q(x=3, y=-1, z=4) touches the Z-axis at the point R(0,0,4) . So, the distance QR is -

QR = $\sqrt{3^2 + 1^2 + 0^2}$

QR = 3.162 units.

PQ = ?

We have,

PQ = $\sqrt{(3.3.441)^2 + (-1 - 4.915)^2 + (4 + 3)^2}$

Hence, PQ = 11.2 units.

6)

a. Express the temperature field $T = 240 + z^2 - \rho^2 \sin 2\Phi$ in cylindrical co-ordinates.
b. Find the density at P(-2,-5,1) if the density is $e^{-z^\wedge 2}(2+\rho^3\cos^2\Phi)$

Solution:

Given, $T = 240 + z^2 - \rho^2 \sin 2\Phi$ --------- (1)

We have relation in cylindrical co-ordinates

$x = \rho \cos\Phi$, $y = \rho \sin\Phi$ and $z = z$

Therefore,

$T = 240 + z^2 - 2\,\rho \cos\Phi.\ \rho \sin\Phi$

$T = 240 + z^2 - \rho^2 \sin 2\Phi$

Given, density = $e^{-z}(2+\rho^3\cos^2\theta)$

At point P(-2,-5,1)

In cylindrical co-ordinates

$P = \sqrt{(-2)^2 + (-5)^2} = 5.385$ units

$\Phi = \tan^{-1}(-5/2) = 68.198^0$

Therefore, point P can be re-written as $P(\rho=5.385, \Phi=68.198^0, z=1)$

So, density = $e^{-1}[2+(5.385)^3 \cos^2 68.19]$

Hence, density = 8.467

7) a. Express the vector field $\mathbf{W} = (x-y)\mathbf{a}_y$ in cylindrical co-ordinates.
   b. Given the field $\mathbf{F}$ in Cartesian co-ordinates if $F = \rho \cos\Phi \, \mathbf{a}_\rho$

Solution:
Given, vector field, $\mathbf{W} = (x-y)\mathbf{a}_y$
In cylindrical co-ordinates, we have –
$x=\rho\cos\Phi$
$y=\rho\sin\Phi$
$z=z$
Also, we have –
$\mathbf{a}_y = \sin\Phi \, \mathbf{a}_\rho + \cos\Phi \, \mathbf{a}_\Phi$
So, the given field can be written as-
$\mathbf{W} = (\rho\cos\Phi - \rho\sin\Phi)(\sin\Phi \, \mathbf{a}_\rho + \cos\Phi \, \mathbf{a}_\Phi)$

Here
$\mathbf{F} = \rho \cos\Phi \, \mathbf{a}_\rho$
In Cartesian co-ordinates relation is given as –
$x=\rho\cos\Phi$
$y=\rho\sin\Phi$
$z=z$
Also, we have –
$\mathbf{a}_\rho = \cos\Phi \, \mathbf{a}_x + \sin\Phi \, \mathbf{a}_y$

so, $\mathbf{F}$ can be written as –
$\mathbf{F} = x(\cos\Phi \, \mathbf{a}_x + \sin\Phi \, \mathbf{a}_y)$

$$= x\left(\frac{x}{\sqrt{x^2+y^2}}\mathbf{a}_x + \frac{y}{\sqrt{x^2+y^2}}\mathbf{a}_y\right)$$

$$= \frac{x}{\sqrt{x^2+y^2}}(x\mathbf{a}_x + y\mathbf{a}_y)$$

8) Given P($r=6, \theta=110^0, \Phi=125^0$) and Q(3,-1,4) . Find the distance from –

a. Q to the origin
b. P to the y=0 plane
c. PQ

Solution:

Distance OQ = $\sqrt{3^2+(-1)^2+4^2}$ = 5.1 units.

Relation between spherical and Cartesian co-ordinates is –

$x = r \sin\theta \cos\Phi = 6 \sin110 \cos125 = -3.234$

$y = r \sin\theta \sin\Phi = 6 \sin110 \sin125 = 4.618$

$z = r \cos\theta = 6 \cos110 = -2.052$

So, P can be re-written as –

P = (x = -3.234,y=4.618,z=-2.052)

Now, distance from point P to y=0 (XZ) plane is given by the y co-ordinate of point P

Hence, distance to y=0 plane = 4.618 units.

PQ = $\sqrt{(3+3.234)^2+(-1-4.618)^2+(4+2.052)^2}$

Hence, PQ = 10.346 units

9)

a. Express the temperature field $T = 240+z^2-2xy$ in spherical co-ordinates
b. Find the density at P(-2,-5,1) if the density is $re^{-r/2}(5+\cos\theta+\sin\theta \cos\Phi)$

Solution:
We have,
Relation between spherical and Cartesian co-ordinates is –
$x = r \sin\theta \cos\Phi$
$y = r \sin\theta \sin\Phi$
$z = r \cos\theta$

Therefore, $T= 240 + r^2\cos^2\theta – 2(r^2\sin^2\theta \sin\Phi \cos\Phi)$
Or, $T = 240 + r^2(\cos^2\theta - r^2\sin^2\theta \sin2\Phi)$

Here, density = $re^{-r/2}(5+\cos\theta+\sin\theta \cos\Phi)$
At point P (-2,-5,1)
Changing to spherical coordinates
$r = \sqrt{(-2)^2+(-5)^2+1} = 5.477$ units.
$\theta = \cos^{-1}(1/5.477) = 79.479^0$
$\cos\Phi = \dfrac{x}{r\sin\theta} = \dfrac{-2}{5.477\sin 79.48}$
Hence, $\Phi = 111.80^0$
So, density = $5.477\, e^{-5.477/2}(5 + \cos79.48 + \sin79.48 \cos111.8)$
Hence, density = 1.604

10)

a. Express the vector field **W** = (x-y)$\mathbf{a_y}$ in cylindrical co-ordinates
b. Give the field **F** in Cartesian co-ordinates if
   F = r cosΦ$\mathbf{a_r}$

Solution:
Given, vector field **W** = (x-y)$\mathbf{a_y}$
In spherical co-ordinates –
x = r sinθ cosΦ
y = r sinθ sinΦ
z = r cosθ

Also, we have –
$\mathbf{a_y}$ = sinθ sinΦ$\mathbf{a_r}$ + cosθ sinΦ$\mathbf{a_\theta}$ +cosΦ$\mathbf{a_\Phi}$

So, the given field can be written as –
W = (r sinθ cosΦ – r sinθ sinΦ) (sinθ sinΦ$\mathbf{a_r}$ + cosθ sinΦ$\mathbf{a_\theta}$ +cosΦ$\mathbf{a_\Phi}$)
Or, **W** = r sinθ (cosΦ – sinΦ) [sinΦ(sinθ$\mathbf{a_r}$ + cosθ$\mathbf{a_\theta}$) + cosΦ$\mathbf{a_\Phi}$]

Here, **F** = r cosΦ$\mathbf{a_r}$
In Cartesian co-ordinates relation is given as –
x = r sinθ cosΦ
y = r sinθ sinΦ
z = r cosθ
Also, we have –
$\mathbf{a_r}$ = sinθ cosΦ$\mathbf{a_x}$ + sinθ sinΦ$\mathbf{a_y}$ + cosθ$\mathbf{a_z}$

Therefore, **F** can be written as-

$$\mathbf{F} = r\cos\Phi[\sin\theta(\cos\Phi\mathbf{a_x} + \sin\Phi\mathbf{a_y}) + \cos\theta\mathbf{a_z}]$$

$$= r\sin\theta\cos\Phi(\cos\Phi\mathbf{a_x} + \sin\Phi\mathbf{a_y}) + r\cos\theta\cos\Phi\mathbf{a_z}$$

$$= x\left(\frac{x}{\sqrt{x^2+y^2}}\mathbf{a_x} + \frac{y}{\sqrt{x^2+y^2}}\mathbf{a_y}\right) + z\frac{x}{\sqrt{x^2+y^2}}\mathbf{a_z}$$

$$= \frac{x}{\sqrt{x^2+y^2}}[x\mathbf{a_x} + y\mathbf{a_y} + z\mathbf{a_z}]$$

11) Transform $\mathbf{A} = y\mathbf{a_x} + x\mathbf{a_y} + x/\sqrt{x^2+y^2}\ \mathbf{a_z}$ from Cartesian to cylindrical co-ordinates.

Solution:
We have, In Cartesian co-ordinates relation is given as –
$x=\rho\cos\Phi$
$y=\rho\sin\Phi$
$z=z$

Also , $\mathbf{a_x} = \cos\Phi\mathbf{a_\rho} - \sin\Phi\mathbf{a_\Phi}$
$\mathbf{a_y} = \sin\Phi\mathbf{a_\rho} + \cos\Phi\mathbf{a_\Phi}$
$\mathbf{a_z} = \mathbf{a_z}$
$\rho=\sqrt{x^2+y^2}$

Therefore,
$\mathbf{A} = \rho\sin\Phi(\cos\Phi\mathbf{a_\rho} - \sin\Phi\mathbf{a_\Phi}) + \rho\cos\Phi(\sin\Phi\mathbf{a_\rho} + \cos\Phi\mathbf{a_\Phi}) + \rho\cos\Phi/\rho\ \mathbf{a_z}$
Or, $\mathbf{A} = \rho\sin\Phi\cos\Phi\mathbf{a_\rho} - \rho\sin\Phi\sin\Phi\mathbf{a_\Phi} + \rho\cos\Phi\sin\Phi\mathbf{a_\rho} + \rho\cos^2\Phi\mathbf{a_\Phi} + \cos\Phi\mathbf{a_z}$
Or, $\mathbf{A} = \rho\sin2\Phi\mathbf{a_\rho} + \rho\cos2\Phi\mathbf{a_\Phi} + \cos\Phi\mathbf{a_z}$

12) Experss the vector $\mathbf{F}=(x^2 + y^2)\mathbf{a_y} + xz\mathbf{a_z}$ into

a. Cylindrical co-ordinate system at $(6,60^0,-4)$
b. Spherical co-ordinate at Q $(4,30^0,120^0)$

Solution:

Coulomb's Law Electric Field Intensity
Tutorial sheet no-2

1) A two mC positive charge is located in vacuum at P_1 (3,-2,-4) and a 5μC negative charge is at P_2 (1,-4,2) .
a) Find the vector force on the negative charge
b) What is the magnitude of the force on the charge at P_1 ?

Solution:
Given,
Charge Q_1 = 2mC at P_1 (3,-2,-4)
Charge Q_2 = 5μC at P_2 (1,-4,2)
The magnitude of force acting on –ve charge due to positive charge is given by –

F = $\frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2}$ $\mathbf{a_{12}}$ [where $\mathbf{a_{12}}$ is unit vector along $\mathbf{P_1P_2}$]

Or, F = $\frac{2*10^{-3}*(-5)*10^{-6}*9*10^{9}}{4+4+36}$

Or, F = -2.0454 [-ve sign shows that force is attractive]

Now, $\mathbf{P_1P_2}$ = -2$\mathbf{a_x}$ -2$\mathbf{a_y}$ +6$\mathbf{a_z}$
Unit vector of $\mathbf{P_1P_2}$ is –
$\mathbf{a_{12}}$ = -2$\mathbf{a_x}$ -2$\mathbf{a_y}$ +6$\mathbf{a_z}$ / $\sqrt{4+4+36}$ = -0.301$\mathbf{a_x}$ -0.301$\mathbf{a_y}$ +0.905$\mathbf{a_z}$

Now the vector force acting on the –ve charge is given by –
F = -2.0454 (-0.301$\mathbf{a_x}$ -0.301$\mathbf{a_y}$ +0.905$\mathbf{a_z}$)
Hence,
F =0.616$\mathbf{a_x}$ +0.616$\mathbf{a_y}$ – 1.848$\mathbf{a_z}$

2) Calculate **E** at M (3,-4,2) in free space caused by –
a) a charge $Q_1 = 2\mu C$ at P_1 (0,0,0)
b) a charge $Q_2 = 3\mu C$ at P_2 (-1,2,3)
c) a charge $Q_1 = 2\mu C$ at P_1 (0,0,0) and a charge $Q_2 = 3\mu C$ at P_2 (-1,2,3)

Solution:
Given,
charge $Q_1 = 2\mu C$ at P_1 (0,0,0)
charge $Q_2 = 3\mu C$ at P_2 (-1,2,3)

We have to find **E** at point M (3,-4,2)
Now,
$\mathbf{P_1M} = 3\mathbf{a_x} -4\mathbf{a_y} +2\mathbf{a_z}$
Unit vector along $\mathbf{P_1M}$ is
$\mathbf{a_1} = 0.557\mathbf{a_x} -0.742\mathbf{a_y} +0.371\mathbf{a_z}$

The value of **E** at point M due to charge Q_1 is given as –

$$\mathbf{E} = \frac{Q_1}{4\pi\varepsilon_0 r^2}\ \mathbf{a_1}$$

$$= \frac{2*10^{-6}*9*10^{9}}{9+16+25}\, 0.557\mathbf{a_x} -0.742\mathbf{a_y} +0.371\mathbf{a_z}$$

$\mathbf{E_1} = 345.72\mathbf{a_x} -460.55\mathbf{a_y} +230.27\mathbf{a_z}$ V/m

Also, we have to find E at point M(3,-4,2) due to charge Q_2
So, $\mathbf{P_2M} = 4\mathbf{a_x} - 6\mathbf{a_y} - \mathbf{a_z}$
$\mathbf{a_2} = 0.549\mathbf{a_x} – 0.824\mathbf{a_y} – 0.137\mathbf{a_z}$
Therefore,

$$\mathbf{E_2} = \frac{Q_2}{4\pi\varepsilon_0 r^2}\mathbf{a_2} = \frac{3*10^{-6}*9*10^{9}}{53}(0.549\mathbf{a_x} – 0.824\mathbf{a_y} – 0.137\mathbf{a_z})$$

$\mathbf{E_2} = 279.678\mathbf{a_x} -420\mathbf{a_y} -70\mathbf{a_z}$ V/m

Now the total effect of M due to Q_1 and Q_2 is –
$\mathbf{E} = \mathbf{E_1} + \mathbf{E_2}$
$\mathbf{E} = 625.4\mathbf{a_x} – 880.55\mathbf{a_y} +160.27\mathbf{a_z}$

3) Find the total charge inside each of the volumes indicated :
a) $\rho_v = 10z^2 e^{-0.1x} \sin\pi y$, $-1<= x <= 2$, $0<=y<=1$, $3<= z <=3.6$

Solution:
We have
$\rho_v = 10z^2 e^{-0.1x} \sin\pi y$
The total charge inside the given volume is given by –

$$Q = \int_{x=-1}^{2}\int_{y=0}^{1}\int_{z=3}^{3.6} 10z^2 e^{-0.1x} \sin \pi y dxdydz$$

$$Q = 10\int_{-1}^{2} e^{-0.1x}dx \int_{0}^{1}\sin \pi y dy \int_{3}^{3.6} z^2 dz$$

$$Q = 10\left[\frac{e^{-0.1x}}{-0.1}\right]_{-1}^{2}\left[\frac{-\cos\pi y}{\pi}\right]_{0}^{1}\left[\frac{z^3}{3}\right]_{3}^{3.6}$$

Q = 10*2.864*2/π *6.552
Q = 119.52 C

b) $\rho_v = 4xyz^2$, $0<= \rho <=2$, $0<= \phi <=\pi/2$, $0<= z <=3$

Solution:
We have,
We have relation in cylindrical co-ordinates
$x = \rho \cos\Phi$, $y = \rho \sin\Phi$ and $z = z$
Therefore,
$\rho_v = 4 \rho^2 \sin\phi \cos\phi z^2$
$= 2 \rho^2 \sin2\phi z^2$

Now, total charge given by ρ_v is

$$Q = \int_{\rho=0}^{2}\int_{\phi=0}^{\pi/2}\int_{z=0}^{3} 2\rho^2 \sin 2\phi z^2 \rho d\rho d\phi dz$$

$$Q = 2\int_{0}^{2}\rho^3 d\rho \int_{0}^{\pi/2}\sin 2\phi d\phi \int_{0}^{3} z^2 dz$$

$$Q = 2\left[\frac{\rho^4}{4}\right]_{0}^{2}\left[\frac{-\cos2\phi}{2}\right]_{0}^{\pi/2}\left[\frac{z^3}{3}\right]_{0}^{3}$$

Q = 2*4*1*9
Q = 72 C

c) $\rho_v = 3\pi \cos^2\phi / [2r^2(r^2+1)]$, universe

Solution:

Total charge is given by –

$$Q = \int_{r=0}^{\infty}\int_{0}^{\pi}\int_{0}^{2\pi} \rho_v r^2 \sin\theta dr d\theta d\phi$$

$$Q = \frac{3\pi}{2}\int_{0}^{\infty}\frac{1}{r^2+1}dr \int_{0}^{\pi}\cos^2\theta \sin\theta d\theta \int_{0}^{2\pi}\cos^2\phi d\phi$$

$$Q = \frac{3\pi}{2}\left[\tan^{-1} r\right]_0^{\infty}\left[\frac{-\cos^3\theta}{3}\right]_0^{\pi}\left[\frac{\sin 2\phi}{2*2}+\frac{\phi}{2}\right]_0^{2\pi}$$

Q = 3π/2 *π/2 * (1/3+1/3) * 2π/2

Q = 15.5 C

4) An infinitely long uniform line charge is located at y=3, z=5 if ρ_L = 30nC/m, Find **E** at:
a) the origin
b) $P_B(0,6,1)$
c) $P_C(5,6,1)$

Solution:
Point of projection from origin to y=3, z=5 is (0,3,5)
$\mathbf{R_{12}} = -3\mathbf{a_y} - 5\mathbf{a_z}$
$|\mathbf{R_{12}}| = \sqrt{34}$
We have
$\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\frac{R_{12}}{|R_{12}|} = -47.6\mathbf{a_y} - 79.3\mathbf{a_z}$ V/m

$\mathbf{R_{12}} = 3\mathbf{a_y} - 4\mathbf{a_z}$
$|\mathbf{R_{12}}| = 5$, $\rho = 5$
We have
$\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\frac{R_{12}}{|R_{12}|} = 64.7\mathbf{a_y} - 86.3\mathbf{a_z}$ V/m

$\mathbf{R_{12}} = 5\mathbf{a_x} + 3\mathbf{a_y} - 4\mathbf{a_z}$
$|\mathbf{R_{12}}| = \sqrt{50}$, $\rho = \sqrt{50}$

We have
$\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\frac{R_{12}}{|R_{12}|} = 53.92\mathbf{a_x} + 32.34\mathbf{a_y} - 43.12\mathbf{a_z}$

5) Four infinite long uniform sheets of charge are located as follows : 20pC/m^2 at y=7, -8pC/m^2 at y=3, 6pC/m^2 at y=-1, and -18pC/m^2 at y=-4. Find **E** at the point :

a) $P_A(2,6,-4)$
b) $P_B(0,0,0)$
c) $P_C(-1,1,5)$
d) $P_D(10^6, 10^6, 10^6)$

Solution :
We have,
$\mathbf{E} = \mathbf{E_{20}} + \mathbf{E_{-8}} + \mathbf{E_6} + \mathbf{E_{-18}}$

$$= \frac{\rho_{20}}{2\varepsilon_0} a_{n20} + \frac{\rho_{-8}}{2\varepsilon_0} a_{n.(-8)} + \frac{\rho_6}{2\varepsilon_0} a_{n6} + \frac{\rho_{-18}}{2\varepsilon_0} a_{n(-18)}$$

$= -2.26\mathbf{a_y}$ V/m

$$\mathbf{E} = \frac{10^{-12}}{2\varepsilon_0}\left[20(-a_y) - 8a_y + 6a_y - 18a_y\right]$$

$= -1.355\mathbf{a_y}$

$$\mathbf{E} = \frac{10^{-12}}{2\varepsilon_0}\left[36a_y\right] = 2.03\mathbf{a_y}$$

$$\mathbf{E} = \frac{10^{-12}}{2\varepsilon_0}\left[20(a_y) - 8a_y + 6a_y - 18a_y\right] = 0$$

6) Obtain the equation of the streamline that passes through the point P(-2,7,10) in the field **E** = :

a) $2(y-1)\mathbf{a_x} + 2x\mathbf{a_y}$

b) $e^y\mathbf{a_x} + (x-1)e^y\mathbf{a_y}$

Solution:

$E_x = 2(y-1)$
$E_y = 2x$
We have,

$$\frac{E_x}{dx} = \frac{E_y}{dy}$$

Or, $\frac{2(y-1)}{dx} = \frac{2x}{dy}$

Or, $(y-1)dy = xdx$
By integration we get –
$y^2/2 - y = x^2/2 + c$
or, $y^2 - 2y = x^2 + c$
Putting x=-2, y=7 in above equation we get
c=31
Hence, $(y-1)^2 - x^2 = 32$

$E_x = e^y$
$E_y = (x-1)\, e^y$
We have,

$$\frac{E_x}{dx} = \frac{E_y}{dy}$$

$dy = (x-1)dx$
By integrating we get –
$x^2/2 + x = y + c$
Or, $x^2 + 2x = 2y + 2c$
Putting x = -2 , y= 7 , we get
c = -7
hence, $2y - (x+1)^2 = 13$

7) A 25μC point charge is located at the origin. Calculate the electric flux passing through :
a) that portion of the sphere r=20cm bounded by θ = 0 and π , ϕ = 0 and π/2
b) the closed surface ρ = 0.8m, z = ±0.5m
c) the plane z = 4m

Solution :

$$\psi = Q = \oint D.ds = \int \frac{Q}{4\pi r^2} \hat{a}_r . r^2 \sin\theta d\theta d\phi \hat{a}_r$$

$$= \int_0^{\pi} \int_0^{\pi/2} \frac{25*10^{-6}}{4\pi} * \sin\theta d\theta d\phi = 6.25 * 10^{-6} \text{ C}$$

Hence, ψ = 6.25μC

8) Find **D** (in Cartesian co-ordinates) at P(6,8,-10) caused by :
a) a point charge of 30mC at the origin
b) a uniform line charge ρ_L = 40μC/m on the z axis
c) a uniform surface charge density ρ_S = 57.2μC/m^2 on the plane x=9

Solution:
$\mathbf{R_{12}} = 6\mathbf{a_x} + 8\mathbf{a_y} - 10\mathbf{a_z}$
$|\mathbf{R_{12}}| = \sqrt{200}$
We have ,
$\mathbf{D} = \dfrac{Q}{4\pi|R_{12}|^2} * \dfrac{R_{12}}{|R_{12}|} = 5.064\mathbf{a_x} + 6.75\mathbf{a_y} - 8.44\mathbf{a_z}$ μC/m^2

$\mathbf{R_{12}} = 6\mathbf{a_x} + 8\mathbf{a_y}$
$|\mathbf{R_{12}}| = 10$
We have ,
$\mathbf{D} = \dfrac{420}{2\pi * 10^2} * \dfrac{R_{12}}{|R_{12}|} = 0.382\mathbf{a_x} + 0.509\mathbf{a_y}$ μC/m^2

$\mathbf{R_{12}} = 6\mathbf{a_x}$
$|\mathbf{R_{12}}| = 6$
$\mathbf{D} = \dfrac{\rho_s}{2} * a_r = 28.6\mathbf{a_x}$ μC/m^2

9) Let $\mathbf{D} = r\mathbf{a_r} /3$ nC/m^2 in free space. Find :
a) $\mathbf{E}$ at r = 0.23m
b) The total charge within the sphere r = 0.2m
c) The total electric flux leaving the sphere r=0.3m

Solution:
We have,
$\mathbf{D} = \varepsilon_0 E$

Or, $\mathbf{E} = \frac{1}{\varepsilon_0} D = 8.66$ V/m

$$Q = \int D.ds = \int r^3 /3 * \sin\theta d\theta d\phi = \frac{0.008}{3}\int_0^{\pi}\int_0^{2\pi} \sin\theta d\theta d\phi$$

= 3.35*10^{-2} nC
= 33.5 pC

$$\Psi = \oint D.ds == \frac{0.027}{3}\int_0^{\pi}\int_0^{2\pi} \sin\theta d\theta d\phi = 113.1 \text{ pC}$$

10) Find the total electric flux leaving the spherical surface $r = 2.5m$ given the charge configuration :

a) $Q = 2^{-x^2}$ nC on the x-axis at $x=0, \pm1,\pm2\ldots\ldots m$

b) A line charge $\rho_L = 1/(z^2-1)$ nC/m on the z-axis

c) A surface charge $\rho_S = 1/(x^2 + y^2 + z^2)$

Solution :

$Q = 2^{-x^2}$ nC at $x = 0,\pm1,\pm2\ldots\ldots m$

$\Psi = \psi_0 + \psi_{-1} + \psi_1 + \psi_2 + \psi_{-2}$

$\psi_0 = 1$

$\psi_{-1} = \psi_1 = 0.5$

$\psi_2 = \psi_{-2} = 0.0625$

Hence, $\psi = 2.25$ nC

11) Surface charge densities of 200-50 and are located at r= and 7cm respectively Find D at r = :

a) 2cm
b) 4cm
c) 6cm
d) Find if D=0 at r=7.32cm

Solution:

12) Let $\mathbf{D} = y^2z^3\mathbf{a_x} - 2xyz^3\mathbf{a_y} + 3xy^2z^2\mathbf{a_z}$ pC/m^2 in free space. Find the

a) total electric flux passing through the surface x=3, 0<=y<=2, 0<=z<=1 in a direction away from the origin.
b) Magnitude of **E** at P(3,2,1) .
c) Total charge contained in an incremental sphere having a radius of 2μm centered at P(3,2,1)

Solution :

We have,

$\Psi = \oint D.ds$ & for $\mathbf{a_x}$ direction

$\mathbf{D_x} = y^2z^3\mathbf{a_x}$

$\mathbf{dS} = dydz\mathbf{a_x}$

$$\psi = \oint y^2z^3dydz = \int_0^2\int_0^1 y^2z^3dydz = 0.667 \text{ pC}$$

we have,

$$\mathbf{E} = \frac{1}{\varepsilon_0}D$$

At P(3,2,1), $D = 4\mathbf{a_x} - 12\mathbf{a_y} + 36\mathbf{a_z}$

Hence, $E = (4\mathbf{a_x} - 12\mathbf{a_y} + 36\mathbf{a_z})/\varepsilon_0$

Or, $E = 0.45\mathbf{a_x} - 1.3\mathbf{a_y} + 4.066\mathbf{a_z}$ V/m

$$|\mathbf{E}| = \sqrt{0.45^2 + (-1.34)^2 + 4.066^2} = 4.31$$

13) consider a co-axial cable having an inner radius of 1mm, outer radius 4mm and length 50cm. Let the space between conductors in filled with air. The total charge on the inner conductor is 30 nC. Find

a) the charge density on the inner conductor
b) the charge density on the outer conductor
c) **D** and
d) E

Solution:
We have,
$Q_{inner} = 2\pi a\rho_s L$
Or, $\rho_s = 9.55 nC/m^2$
Also,
$Q_{outer} = Q_{inner}$
Or, $\rho_{souter} = - a/b\ \rho_{sinner}$
= 1*10-3/(4*10-3)*9.55
= -2.3nC/m^2

D = $\frac{a\rho_s}{\rho} * a_\rho$ = 9.55/ρ $\mathbf{a_\rho}$ nC/m^2

E = $\frac{a\rho_s}{\varepsilon_0 \rho} a_\rho$ = 1079/ρ $\mathbf{a_\rho}$ V/m

14) Four infinite uniform sheet of charge are located as follows 20pc/m^2 at y=7, 8pC/m^2 at y=-1, -18 pC/m^2 at y=-4, Find **E** at points

a) A(2,6,-4)
b) B(0,0,0)
c) C(-1,-1,1.5)
d) D(10^6,10^6,10^6)

Solution :
We have,

$\mathbf{E} = \mathbf{E}_{20} + \mathbf{E}_{-8} + \mathbf{E}_{6} + \mathbf{E}_{-18}$

$$= \frac{\rho_{20}}{2\varepsilon_0} a_{n20} + \frac{\rho_{-8}}{2\varepsilon_0} a_{n.(-8)} + \frac{\rho_6}{2\varepsilon_0} a_{n6} + \frac{\rho_{-18}}{2\varepsilon_0} a_{n(-18)}$$

$= -2.26\mathbf{a_y}$ V/m

$$\mathbf{E} = \frac{10^{-12}}{2\varepsilon_0}\left[20(-a_y) - 8a_y + 6a_y - 18a_y\right]$$

$= -1.355\mathbf{a_y}$

$$\mathbf{E} = \frac{10^{-12}}{2\varepsilon_0}\left[36a_y\right] = 2.03\mathbf{a_y}$$

$$\mathbf{E} = \frac{10^{-12}}{2\varepsilon_0}\left[20(a_y) - 8a_y + 6a_y - 18a_y\right] = 0$$

15) In free space let Q_1 = 10nC be at $P_1(0,-4,0)$ & Q_2 =20nC at $P_2(0,0,4)$

a) find **E** at the origin
b) where should the 30 nC point charge be located so that **E** = 0 at the origin

solution :

$\mathbf{R_{12}} = 4\mathbf{a_y}$

$|\mathbf{R_{12}}| = 4$

$\mathbf{R_{12}^{1}} = -4\mathbf{a_z}$

$|\mathbf{R_{12}^{1}}| = 4$

We have,

$$\mathbf{E_1} = \frac{q_1}{4\pi\varepsilon_0|R_{12}|^2} * \frac{R_{12}}{|R_{12}|} = 5.617\ \mathbf{a_y}$$

$$\mathbf{E_2} = \frac{q_2}{4\pi\varepsilon_0|R_{12}^{1}|^2} * \frac{R_{12}^{1}}{|R_{12}^{1}|} = -11.235\mathbf{a_z}$$

Hence, $\mathbf{E}=\mathbf{E_1}+\mathbf{E_2} = 5.617\mathbf{a_x} - 11.235\mathbf{a_z}$ V/m

16) A point charge $Q_a = 1\mu C$ is at A(0,0,1) & $Q_b = -1\mu C$ is at B(0,0,-1) then find $\mathbf{E_r}$, $\mathbf{E_\theta}$, $\mathbf{E_\phi}$ at P(1,2,3)

Solution:
We have,
$\mathbf{E} = E_r\,\mathbf{a_r} + E_\theta\,\mathbf{a_\theta} + E_\phi\mathbf{a_\phi}$ where $\theta = 36.69^0$ & $\phi = 63.43^0$
$\mathbf{R_{12}} = \mathbf{a_x} + 2\mathbf{a_y} + 2\mathbf{a_z}$
$|\mathbf{R_{12}}| = 3$
$\mathbf{R_{12}^1} = \mathbf{a_x} + 2\mathbf{a_y} + 4\mathbf{a_z}$
$|\mathbf{R_{12}^1}| = \sqrt{21}$
$\mathbf{E_a} = \frac{Q_a}{4\pi\varepsilon_0|R_{12}|^2} * \frac{R_{12}}{|R_{12}|} = 332.88\mathbf{a_x} + 665.75\mathbf{a_y} + 665.75\mathbf{a_z}$

$\mathbf{E_b} = \frac{Q_b}{4\pi\varepsilon_0|R_{12}|^2} * \frac{R_{12}}{|R_{12}|} = -93.395\mathbf{a_x} - 186.78\mathbf{a_y} - 373.58\mathbf{a_z}$

Hence, $\mathbf{E} = \mathbf{E_a} + \mathbf{E_b} = 239.48\mathbf{a_x} + 478.98\mathbf{a_y} + 292.18\mathbf{a_z}$

Let
$\mathbf{E_r} = E.\mathbf{a_r} = 239.48 * \sin 36.69 * \cos 63.43 + 478.98 \sin 36.69 * \sin 63.43 + 292.18 \cos 36.69 = 426.3$
$\mathbf{E_\theta} = E.\mathbf{a_\theta} = 239.48 \cos 36.69 * \cos 63.43 + 478.98 \cos 36.69 * \sin 63.43 - 292.98 \sin 36.69$
$= 340.35$
$\mathbf{E_\phi} = E.\mathbf{a_\phi} = -239.48 \sin 63.43 + 478.98 \cos 63.43 + 0$
$= 0.0552$
Hence $\mathbf{E} = 426.3\mathbf{a_r} + 340.35\mathbf{a_\theta} + 0.552\mathbf{a_\phi}$

17) find the total charge within the region for which ρv = 40xyz C/m3 where

a) 0<=x,y,z<=2

b) x=0, y=0, 0<=2x+3y<=10, 0<=z<=2

solution:

We have,

$$Q = \int_v \rho_v dv = \int_0^2\int_0^2\int_0^2 40xyzdxdydz = 320 \text{ C}$$

Again,

$$Q = \int_v \rho_v dv = \int_0^5\int_0^{10/3}\int_0^2 40xyzdxdydz = 5555.56 \text{ C}$$

18) The volume charge density is given by $\rho_v = 10e^{-1000\rho}e^{-100z}$ C/m^3 . Find

a) maximum value of ρ_v where 0<=ρ<=0.01 m, 0<=ϕ<=2π, 0<=z<=0.01 m

b) find the total charge contained in the first quadrant where x,y,z are positive

c) find b if the total charge found in the volume 0<=ρ<=b, 0<=ϕ<=π/2, z>=0, is half the value found in b) above:

Solution :

$\frac{d\rho_v}{d\rho} = -10000e^{-1000\rho}\, e^{-100z}$

For maxima or minima

$\frac{d\rho_v}{d\rho} = 0$

Or, $-10000e^{-1000\rho}\, e^{-100z} = 0$

Either $\rho = \infty$ or $z = \infty$

Therefore,

$\rho_{vmax} = 10e^{-\infty}.\, e^{-\infty} = 10$ C/m^3

$$Q = \int_v \rho_v dv = \int_0^{0.01}\int_0^{\pi/2}\int_0^{0.01} 10\rho e^{-1000\rho} e^{-100z}\, d\rho d\phi dz$$

Integrating by parts

$Q = \pi/2[11e^{-20} - 12e^{-10} + 1]/1000000 = 157.08$ nC

$$Q = \int_v \rho_v dv = \int_0^{b}\int_0^{\pi/2}\int_0^{0.01} 10\rho e^{-1000\rho} e^{-100z}\, d\rho d\phi dz$$

Now by the question

Q = ½ Q (b)

Or, $1.5708*10^{-7.} = \pi/2[-1000b\, e^{-1000b} - e^{-1000b}]\, [-e^{-10} + 1] / 10000000$

Or, b = 1.67835

19) let $\rho_v = (x + 2y + 3z)$ C/m3 in a cubical region, 0<=x,y,z<=1 mm and $\rho_v = 0$ outside the cube.

a) what is the total charge contained within this cube
b) set up the volume integral that will give E(x,0,0) for x>0 mm not integrate

Solution:

The total charge contained within the cube = Q = $\int_v \rho_v dv =$

Or, Q = $\int_0^1\int_0^1\int_0^1 (x+2y+3z)dxdydz$

Or, Q = 3pC

20) Volume charge density is given as $\rho_v = 10^{-5}e^{-100r}\sin\theta$ C/m^3 for 0<=r<=1 cm & ρ_v = 0 for r>1 cm find **E** at r=1cm , $\theta = 90^0$ and ϕ =0 by taking in terms of a point charge.

Solution:

Q = $\int_v \rho_v dv = \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} 10^{-5} e^{-100r} \sin\theta r^2 dr d\theta d\phi$

Q = 1.585 $*10^{-11}$ C

E at r=1

E = $\frac{Q}{4\pi\varepsilon_0 r^2}$ $\mathbf{a_r}$ = $\frac{9.86*10^{-11}}{4\pi(8.854*10^{-12})}$ $\mathbf{a_x}$

Or, **E** = 0.142$\mathbf{a_x}$